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Phase transition in a non-translationally invariant spherical model

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Abstract. A non-translationally invariant spherical model, in which only a finite number of spins interact, is solved exactly. The model exhibits a phase transition in a non-zero uniform field, without spontaneous magnetization. The anomalous transition is attributed to the finite number of interacting spins taking on abnormally large values of order $N^{1/2}$ without contributing to the magnetization. The free energy of the model can be obtained from a spherical limit ($n \to \infty$) of a corresponding n-vector model. In zero field the free energy is of the Curie–Weiss (or mean-field) spherical form. The Curie–Weiss form can only be maintained in a field by admitting a non-uniform field of order $N^{1/2}$. This modified spherical model is also accessible from an $n \to \infty$ limit of a corresponding n-vector model.

1. Introduction and summary

In treating the spherical model (Berlin and Kac 1952) it has become customary (Joyce 1972) to assume translationally invariant interactions. In this paper we solve a non-translationally invariant spherical model exactly. The solution allows a direct study of the resulting anomalous phase transition.

The model consists of a set of N spherical spins $-\infty < x_i < \infty$, $i = 1, 2, \ldots, N$, with the non-translationally invariant interaction energy

$$\mathscr{H} = -\sum_{1 \leq i < j \leq k} \rho_{ij} x_i x_j - H \sum_{i=1}^{N} x_i, \qquad 2 \leq k < N \tag{1.1}$$

where k is a fixed integer and the spins are subject to the spherical constraint

$$\sum_{i=1}^{N} x_i^2 = N. \tag{1.2}$$

The partition function is ($\beta = 1/k_B T$)

$$Z_N(\beta, H) = A_N^{-1} \int \cdots \int_{\sum_{i=1}^{N} x_i^2 = N} d^N x \exp\left(\tfrac{1}{2}\beta \sum_{i,j=1}^{k} \rho_{ij} x_i x_j + \beta H \sum_{i=1}^{N} x_i\right), \tag{1.3}$$

where we have set $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = 0$. The normalization constant is given by

$$A_N = (2\pi)^{N/2} N^{(N-1)/2} / \Gamma(N/2) \tag{1.4}$$

and for later use we note that by Stirling's formula

$$\lim_{N\to\infty} N^{-1} \ln A_N = (\ln 2\pi + 1)/2. \tag{1.5}$$

For definiteness in our calculation we will restrict ourselves to the ferromagnetic case ($\rho_{ij} \geqslant 0$) with ρ a cyclic $k \times k$ matrix. The maximum eigenvalue λ_1 of ρ is then given by

$$\lambda_1 = \sum_{j=1}^{k} \rho_{ij} \tag{1.6}$$

independent of the value of i.

The case $k=2$, $H=0$ of (1.1) was discussed by Lieb and Thompson (1969), who showed that the free energy is identical to that of the Curie–Weiss (or mean-field) spherical model, in which all spins interact equally with one another. We show that this is not the case for the model (1.1) in non-zero field ($H \neq 0$).

The motivation for considering the model (1.1) comes from a recent paper (Pearce and Thompson 1976), in which we made an incomplete comparison with the corresponding n-vector model in the spherical limit ($n \to \infty$). The main result of this previous paper was that the free energy ψ_n' of a certain anisotropic n-vector model is equal to the *Curie–Weiss* spherical model free energy in the spherical limit ($n\to\infty$). To be more precise, we introduce the n-vector model defined by the Hamiltonian

$$\mathscr{H}_n' = -\sum_{\alpha=1}^{k} \sum_{1\leqslant i<j\leqslant N} \rho_{ij} S_{i\alpha} S_{j\alpha} - (n/k)^{1/2} H \sum_{\alpha=1}^{k} \sum_{i=1}^{N} S_{i\alpha} \tag{1.7}$$

and partition function

$$Q_N^n(\beta, H) = (A_n)^{-N} \int \cdots \int_{\|\boldsymbol{S}_i\| = n^{1/2}} \exp(-\beta \mathscr{H}_n')\, \mathrm{d}^N \boldsymbol{S}, \tag{1.8}$$

where the n-dimensional spins have norm

$$\|\boldsymbol{S}_i\| = \left(\sum_{\alpha=1}^{n} S_{i\alpha}^2\right)^{1/2} = n^{1/2}. \tag{1.9}$$

Our previous result can now be stated as

$$\lim_{n\to\infty} \beta\psi_n'(\beta, H) = -\lim_{\substack{N,n\to\infty \\ (k\ \text{fixed})}} (Nn)^{-1} \ln Q_N^n(\beta, H)$$

$$= \min_{r\geqslant 0} \tfrac{1}{2}[\![\nu r^2 - (1+4z^2)^{1/2} + 1 + \ln\{\tfrac{1}{2}[1+(1+4z^2)^{1/2}]\}]\!] \tag{1.10}$$

where

$$z = \nu r + \beta H, \tag{1.11}$$

and

$$\nu = \beta \lim_{N\to\infty} N^{-1} \sum_{i,j=1}^{N} \rho_{ij} < \infty. \tag{1.12}$$

Here ρ is an $N \times N$ matrix, assumed to be cyclic.

In zero field (1.10) reduces to the free energy of the Lieb and Thompson (1969) model,

$$\lim_{n\to\infty} \beta\psi_n(\beta, 0) = \begin{cases} \frac{1}{2}(-\nu+1+\ln \nu) & \nu>1 \\ 0 & \nu<1, \end{cases} \tag{1.13}$$

in accord with the observed equivalence (Hikami 1974) of this model and the corresponding limiting anisotropic n-vector model. In the present context of non-zero field it is important to stress that, following Moore *et al* (1974), an *anisotropic field* was considered in our previous paper (i.e. in (1.7)). This necessitated taking the external field H of order $n^{1/2}$ to achieve a field contribution to the free energy. Here we show that if the allied, and perhaps more natural, n-vector Hamiltonian

$$\mathcal{H}_n = -\sum_{\alpha=1}^{k} \sum_{1\leq i<j\leq N} \rho_{ij} S_{i\alpha} S_{j\alpha} - H \sum_{\alpha=1}^{n} \sum_{i=1}^{N} S_{i\alpha}, \tag{1.14}$$

with an *isotropic field*, is considered in place of (1.7), then the pathological spherical model (1.1) obtains in the spherical limit ($n\to\infty$). It follows as a consequence that some care must be exercised in proceeding to the spherical limit ($n\to\infty$) for systems with anisotropic interactions.

In summary, the thermodynamic quantities for model (1.1) are obtained in § 2. We find that, under the assumed conditions on ρ, the free energy $\psi(\beta, H)$ and the magnetization $m(\beta, H)$ are given respectively by

$$-\beta\psi(\beta, H) = \lim_{\substack{N\to\infty \\ (k \text{ fixed})}} N^{-1} \ln Z_N(\beta, H) = \tfrac{1}{2}[z_s - \ln z_s - 1 + (\beta^2 H^2/z_s)] \tag{1.15}$$

and

$$m(\beta, H) = -\frac{\partial\psi}{\partial H} = \frac{\beta H}{z_s}, \tag{1.16}$$

where

$$z_s(\beta, H) = \begin{cases} \frac{1}{2} + (\frac{1}{4} + \beta^2 H^2)^{1/2} & T > T_c(H) \\ \beta\lambda_1 & T < T_c(H). \end{cases} \tag{1.17}$$

The critical temperature $T_c(H)$ is found to depend on the field according to

$$k_B T_c(H) = \lambda_1 - H^2/\lambda_1, \qquad |H| < \lambda_1. \tag{1.18}$$

When $|H| \geq \lambda_1$ there is no phase transition.

In order to elucidate the pathological nature of the phase transition we investigate various order parameters in § 3. In particular, it is shown that the singular free energy without spontaneous magnetization (from (1.16)) is due to the fact that the single spin averages $\langle x_i \rangle$ in the 'low-temperature' region are of order $N^{1/2}$ for $1 \leq i \leq k$ and of order unity for $k+1 \leq i \leq N$, while all are of order unity in the high-temperature region: the shift to order $N^{1/2}$ values reflecting singular behaviour without contributing to the spontaneous magnetization.

Finally, contact is made with the n-vector model in § 4, where we show that the free energy (1.15), of the anisotropic spherical model (1.1), can be obtained from the anisotropic n-vector model (1.14) in the spherical limit ($n\to\infty$). To complete the identification of limiting n-vector models with spherical models, we further show that the limiting free energy (1.10) of the anisotropic n-vector model (1.7) with an anisotropic field is obtained from the anisotropic spherical model with interaction energy

$$\mathscr{H}' = -\sum_{1\leq i<j\leq k} \rho_{ij}x_i x_j - (N/k)^{1/2} H \sum_{i=1}^{k} x_i, \tag{1.19}$$

by admitting a field, dependent on the number of spins N, and acting only on the k interacting spins.

2. The thermodynamic functions

To evaluate the partition function (1.3) we write the spherical constraint as a δ function in the integrand to obtain

$$Z_N(\beta, H) = 2N^{1/2}A_N^{-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{d}^N x\,\delta\Big(\sum_{i=1}^{N} x_i^2 - N\Big)\exp\Big(\tfrac{1}{2}\beta\sum_{i,j=1}^{k}\rho_{ij}x_i x_j + \beta H\sum_{i=1}^{N} x_i\Big). \tag{2.1}$$

Using the integral representation

$$\delta(x) = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \exp(-sx)\,\mathrm{d}s \tag{2.2}$$

for the δ function we then obtain (Berlin and Kac 1952)

$$Z_N(\beta, H) = \frac{2N^{1/2}A_N^{-1}}{2\pi\mathrm{i}}\int_{\alpha-\mathrm{i}\infty}^{\alpha+\mathrm{i}\infty} \exp(Ns)Q_N(\beta, H; s)\,\mathrm{d}s \tag{2.3}$$

where

$$Q_N(\beta, H; s) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{d}^N x \exp\Big(-s\sum_{i=1}^{N} x_i^2 + \tfrac{1}{2}\beta\sum_{i,j=1}^{k}\rho_{ij}x_i x_j + \beta H\sum_{i=1}^{N} x_i\Big) \tag{2.4}$$

and α is taken large enough so that the contour in (2.3) is to the right of all singularities of $Q_N(\beta, H; s)$.

The auxiliary function $Q_N(\beta, H; s)$ is the mean-spherical partition function. Direct evaluation yields

$$Q_N(\beta, H; s) = (\pi/s)^{(N-k)/2}\,\mathrm{e}^{\beta^2 H^2 (N-k)/4s}$$

$$\times\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{d}^k x \exp\Big(-\sum_{i,j=1}^{k}(s\delta_{ij} - \tfrac{1}{2}\beta\rho_{ij})x_i x_j + \beta H\sum_{i=1}^{k} x_i\Big) \tag{2.5}$$

$$= \pi^{N/2}s^{-(N-k)/2}\prod_{r=1}^{k}(s - \tfrac{1}{2}\beta\lambda_r)^{-1/2}\exp\Big[\tfrac{1}{4}\beta^2 H^2\Big(\frac{k}{s-\frac{1}{2}\beta\lambda_1} + \frac{N-k}{s}\Big)\Big] \tag{2.6}$$

provided

$$\operatorname{Re} s > \tfrac{1}{2}\beta\lambda_1. \tag{2.7}$$

Here λ_r are the k eigenvalues of the matrix ρ introduced by the diagonalization of the quadratic form in the exponent of (2.5).

The integral (2.3) can now be evaluated by the method of steepest descents (Berlin and Kac 1952) giving for the free energy (see (1.5))

$$-\beta\psi(\beta, H) = \lim_{N\to\infty} N^{-1} \ln Z_N(\beta, H) \tag{2.8}$$

$$= \tfrac{1}{2}\left(z_s - \ln z_s - 1 + \frac{\beta^2 H^2}{z_s} \right) \tag{2.9}$$

where the saddle point $s = \frac{1}{2}z_s$ is determined from

$$1 = \frac{1}{z_s} + \frac{\beta^2 H^2}{z_s^2}, \tag{2.10}$$

that is,

$$z_s(\beta, H) = \tfrac{1}{2} + (\tfrac{1}{4} + \beta^2 H^2)^{1/2}. \tag{2.11}$$

The function $Q_N(\beta, H; s)$ is only analytic in the complex plane cut from the branch point $s = \frac{1}{2}\beta\lambda_1$ to $s = -\infty$, so the above saddle point method is only valid if the saddle point occurs to the right of the branch point, i.e.,

$$z_s(\beta, H) > \beta\lambda_1. \tag{2.12}$$

This inequality is satisfied if

$$|H| > \lambda_1 \tag{2.13}$$

or

$$\beta < \beta_c(H) = \frac{\lambda_1}{\lambda_1^2 - H^2}. \tag{2.14}$$

This 'high-temperature' region is separated from the 'low-temperature' region in the temperature–field plane by the critical curve

$$k_B T_c(H) = \lambda_1 - \frac{H^2}{\lambda_1} \qquad (|H| < \lambda_1). \tag{2.15}$$

In the 'low-temperature' region,

$$\beta > \beta_c(H) = \frac{\lambda_1}{\lambda_1^2 - H^2} \qquad |H| < \lambda_1, \tag{2.16}$$

we must evaluate a contour integral. The integral to consider is

$$\frac{2N^{1/2}A_N^{-1}}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{Ns} \pi^{N/2} s^{-N/2} (s - \tfrac{1}{2}\nu)^{-1/2} e^{NB^2/4s} \, ds \tag{2.17}$$

$$= \frac{N^{1/2}\Gamma(N/2)}{2\pi i} \int_{\alpha N - i\infty}^{\alpha N + i\infty} e^z (z - \tfrac{1}{2}\nu N)^{-1/2} z^{-N/2} e^{N^2 B^2/4z} \, dz \tag{2.18}$$

where we have set $\nu = \beta\lambda_1$, $B = \beta H$ and $z = Ns$. This integral can be evaluated by using the inverse Laplace transforms (Erdélyi 1954):

$$\frac{1}{2\pi i} \int_{z_0 - i\infty}^{z_0 + i\infty} e^{tz} (z - a)^{-1/2} \, dz = (\pi t)^{-1/2} e^{at}, \tag{2.19}$$

and

$$\frac{1}{2\pi i}\int_{z_0-i\infty}^{z_0+i\infty} e^{tz}z^{-\mu}\, e^{a^2/4z}\, dz = t^{\mu-1}I_{\mu-1}(at^{1/2})/(\tfrac{1}{2}at^{1/2})^{\mu-1}, \qquad \mu>0. \tag{2.20}$$

Here I_μ is a modified Bessel function of order μ. By the convolution formula the integral (2.18) becomes

$$(N/\pi)^{1/2}\Gamma(N/2)\int_0^1 [(1-t)^{-1/2}\, e^{\frac{1}{2}\nu N(1-t)}t^{\frac{1}{2}N-1}I_{\frac{1}{2}N-1}(NBt^{1/2})/(\tfrac{1}{2}NBt^{1/2})^{\frac{1}{2}N-1}]\, dt. \tag{2.21}$$

This integral can now be evaluated by Laplace's method. We have the asymptotic formula (Pearce and Thompson 1976)

$$\Gamma(N/2)I_{\frac{1}{2}N-1}(Nz)/(\tfrac{1}{2}Nz)^{\frac{1}{2}N-1} \sim \exp[\![N/2\{(1+4z^2)^{1/2}-1-\ln\tfrac{1}{2}[1+(1+4z^2)^{1/2}]\}]\!] \qquad \text{as } N\to\infty. \tag{2.22}$$

Hence we conclude that

$$\begin{aligned}-\beta\psi(\beta,H) &= \lim_{N\to\infty} N^{-1}\ln Z_N(\beta,H)\\ &= \max_{0\leqslant t\leqslant 1} \tfrac{1}{2}[\![\nu(1-t)+\ln t+(1+4\beta^2H^2t)^{1/2}-1-\ln\{\tfrac{1}{2}[1+(1+4\beta^2H^2t)^{1/2}]\}]\!].\end{aligned} \tag{2.23}$$

For $\nu<\frac{1}{2}+(\frac{1}{4}+\beta^2H^2)^{1/2}$, that is in the 'high-temperature' region, the maximum occurs for $t=1$ so that the solution (2.23) is easily seen to reduce to the normal saddle point solution given by (2.9) and (2.11). In the 'low-temperature' region, $\nu>\frac{1}{2}+(\frac{1}{4}+\beta^2H^2)^{1/2}$, the maximum occurs when

$$t=(\nu+\beta^2H^2)/\nu^2 \tag{2.24}$$

and consequently

$$-\beta\psi(\beta,H)=\tfrac{1}{2}\left(\nu-\ln\nu-1+\frac{\beta^2H^2}{\nu}\right). \tag{2.25}$$

Comparison with (2.9) and (2.12) shows that the solution (2.25) can be interpreted as arising from the 'sticking' of the saddle point at $z_s=\nu\equiv\beta\lambda_1$ in the 'low-temperature' region. It is precisely this 'sticking' of the saddle point that is responsible for the break in analyticity of the free energy on the critical curve given by (2.15).

To evaluate the remaining thermodynamic functions, we hold the temperature fixed and write

$$-\beta\psi(z_s,B)=\tfrac{1}{2}\left(z_s-\ln z_s-1+\frac{B^2}{z_s}\right) \tag{2.26}$$

with

$$z_s(B)=\begin{cases}\frac{1}{2}+(\frac{1}{4}+B^2)^{1/2} & T>T_c(H)\\ \nu & T<T_c(H).\end{cases} \tag{2.27}$$

The magnetization is now given by

$$m(z_s, B) = \frac{\mathrm{d}}{\mathrm{d}B}(-\beta\psi) = \frac{\partial}{\partial B}(-\beta\psi) + \frac{\partial}{\partial z_s}(-\beta\psi)\frac{\mathrm{d}z_s}{\mathrm{d}B} = \begin{cases} B/z_s & T > T_c(H) \\ B/\nu & T < T_c(H) \end{cases} \tag{2.28}$$

and the susceptibility by

$$\chi(z_s, B) = \frac{\mathrm{d}m}{\mathrm{d}B} = \frac{\partial m}{\partial B} + \frac{\partial m}{\partial z_s}\frac{\mathrm{d}z_s}{\mathrm{d}B} = \begin{cases} \dfrac{1}{z_s} - \dfrac{B^2}{z_s^2(\frac{1}{4}+B^2)^{1/2}} & T > T_c(H) \\ 1/\nu & T < T_c(H). \end{cases} \tag{2.29}$$

In particular, we remark that there is no spontaneous magnetization and that the susceptibility remains finite with a simple jump discontinuity across the critical curve, except at the 'Curie point', $H = 0$, $k_B T = \lambda_1$, where it has a cusp.

3. Order parameters

The usual thermodynamic functions are not very illuminating as to the mechanism of the phase transition. To elucidate the matter, we will investigate the order of magnitude of the spins as $N \to \infty$.

We compute $\langle x_i \rangle$ and begin by noting that, since ρ is a cyclic matrix,

$$\langle x_i \rangle = \frac{1}{k}\left\langle \sum_{i=1}^{k} x_i \right\rangle, \qquad 1 \leqslant i \leqslant k. \tag{3.1}$$

Next, we introduce the auxiliary function

$$\begin{aligned} Q_N(\beta, H, H'; s) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}^N x \exp\left(-s \sum_{i=1}^{N} x_i^2 + \tfrac{1}{2}\beta \sum_{i,j=1}^{k} \rho_{ij} x_i x_j + \beta H' \sum_{i=1}^{k} x_i + \beta H \sum_{i=k+1}^{N} x_i\right) \\ &= \pi^{N/2} s^{-(N-k)/2} \prod_{r=1}^{k} (s - \tfrac{1}{2}\beta\lambda_r)^{-1/2} \exp\left(\frac{k(\beta H')^2}{4(s - \frac{1}{2}\beta\lambda_1)} + \frac{\beta^2 H^2 (N-k)}{4s}\right). \end{aligned} \tag{3.2}$$

Clearly from (2.3) we can now write

$$\begin{aligned} Z_N(\beta, H)\left\langle \frac{1}{k}\sum_{i=1}^{k} x_i \right\rangle &= \frac{2N^{1/2} A_N^{-1}}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \mathrm{d}s\, \mathrm{e}^{Ns} \frac{1}{k} \frac{\partial}{\partial(\beta H')} Q_N(\beta, H, H'; s)\bigg|_{H'=H} && (3.3) \\ &= \frac{2N^{1/2} A_N^{-1}}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \mathrm{d}s\, \mathrm{e}^{Ns} \frac{\beta H}{2(s - \frac{1}{2}\beta\lambda_1)} Q_N(\beta, H; s). && (3.4) \end{aligned}$$

In proceeding some care must be exercized. The contour integral to be considered now is

$$\begin{aligned} T_\mu &= \frac{2A_N^{-1}}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \mathrm{d}s\, \mathrm{e}^{Ns} \pi^{N/2} s^{-N/2} (s - \tfrac{1}{2}\nu)^{-\mu}\, \mathrm{e}^{NB^2/4s}\, \mathrm{e}^{kB^2/4(s - \frac{1}{2}\nu)} && (3.5) \\ &= \frac{N^{\mu - \frac{1}{2}} \Gamma(N/2)}{2\pi \mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} \mathrm{d}z\, \mathrm{e}^{z} (z - \tfrac{1}{2}\nu N)^{-\mu}\, \mathrm{e}^{kNB^2/4(z - \frac{1}{2}\nu N)} z^{-N/2}\, \mathrm{e}^{N^2 B^2/4z}. && (3.6) \end{aligned}$$

Again by the convolution formula and the inverse Laplace transform (2.20) we have

$$T_\mu = N^{\mu-\frac{1}{2}}\Gamma(N/2)\int_0^1 \mathrm{d}t\, \mathrm{e}^{\frac{1}{2}\nu N(1-t)}$$
$$\times (1-t)^{\mu-1}\frac{I_{\mu-1}((kNB^2)^{1/2}(1-t)^{1/2})}{(\frac{1}{2}(kNB^2)^{1/2}(1-t)^{1/2})^{\mu-1}}t^{\frac{1}{2}N-1}\frac{I_{\frac{1}{2}N-1}(NBt^{1/2})}{(\frac{1}{2}NBt^{1/2})^{\frac{1}{2}N-1}}. \tag{3.7}$$

Laplace's method can now be applied to this integral as before, only now we need the asymptotic formula (Abramowitz and Stegun 1964)

$$I_\mu(N^{1/2}z)\sim(2\pi N^{1/2}z)^{-1/2}\,\mathrm{e}^{N^{1/2}z} \qquad \text{as } N\to\infty, \tag{3.8}$$

for the first modified Bessel function in (3.7).

Returning to (3.1), (3.2) and (3.4) we see that

$$\langle x_i\rangle = \frac{T_{3/2}}{T_{1/2}} \sim \frac{B}{2}\frac{2}{(kNB^2)^{1/2}}N(1-t_0)^{1/2}, \qquad 1\leqslant i\leqslant k, \tag{3.9}$$

where the critical value of t is given, as found previously, by

$$t_0 = \begin{cases} 1 & T>T_c(H) \\ (\nu+\beta^2H^2)/\nu^2 & T<T_c(H). \end{cases} \tag{3.10}$$

We conclude that for $1\leqslant i\leqslant k$

$$\lim_{N\to\infty} N^{-1/2}\langle x_i\rangle = \begin{cases} 0 & T>T_c(H) \\ \left(\dfrac{\nu^2-\nu-\beta^2H^2}{k\nu^2}\right)^{1/2}\operatorname{sgn} H & T<T_c(H). \end{cases} \tag{3.11}$$

It is now clear that the phase transition is characterized by the finite number of interacting spins taking on abnormally large values of order $N^{1/2}$ below the critical temperature. Notwithstanding the magnetization is entirely due to *non-interacting* spins since from (2.28)

$$m = \lim_{N\to\infty}\left\langle\frac{1}{N}\sum_{i=1}^{N}x_i\right\rangle \tag{3.12}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{k}\langle x_i\rangle + \lim_{N\to\infty}\frac{N-k}{N}\langle x_j\rangle \qquad j>k$$

$$= \lim_{N\to\infty}\langle x_j\rangle, \qquad j>k. \tag{3.13}$$

The non-interacting spins are clearly of order unity as $N\to\infty$.

The failure of the k interacting spins to contribute to the magnetization explains the absence of spontaneous magnetization in the model (1.1). Notice, however for $1\leqslant i\leqslant k$,

$$\lim_{H\to 0^+}\lim_{N\to\infty}(k/N)^{1/2}\langle x_i\rangle = \begin{cases} 0 & T>T_c(0) \\ [1-(1/\nu)]^{1/2} & T<T_c(0) \end{cases} \tag{3.14}$$

breaking the zero-field symmetry. The order of taking the limits here cannot be interchanged.

Although the result (3.14) is reminiscent of spontaneous magnetization, it is not connected in any way to the analyticity of the free energy $\psi(\beta, H)$. It does, however, provide a clue on how to rig the field terms to recover the Curie–Weiss free energy, thereby making contact with the $n \to \infty$ limit of the anisotropic n-vector model (1.7). This will be pursued in the next section after first identifying the anisotropic spherical model (1.1) as a limiting n-vector model.

4. The spherical limit

The limiting free energy for the n-vector model,

$$\mathcal{H}_n = -\sum_{\alpha=1}^{k} \sum_{1 \le i < j \le N} \rho_{ij} S_{i\alpha} S_{j\alpha} - H \sum_{\alpha=1}^{n} \sum_{i=1}^{N} S_{i\alpha}, \tag{4.1}$$

can be obtained by a trivial extension of the analysis of Pearce and Thompson (1976). The result is

$$\lim_{n\to\infty} \beta\psi_n(\beta, H) = \min_{r \ge 0} \tfrac{1}{2}[\![\nu r^2 - (1+4z^2)^{1/2} + 1 + \ln\{\tfrac{1}{2}[1+(1+4z^2)^{1/2}]\}]\!] \tag{4.2}$$

where

$$z = (\nu^2 r^2 + \beta^2 H^2)^{1/2}. \tag{4.3}$$

The form of the free energy (4.2) is exactly the same as that found in the anisotropic field case (1.10). The local field z, however, is given, according to the extended analysis, by

$$z = \lim_{N,n\to\infty} \|\nu_N \boldsymbol{r} + n^{-1/2} \beta \boldsymbol{H}\| \tag{4.4}$$

where the k-vector $\boldsymbol{r}$ is the projection of the magnetization onto k-space, the n-vector field $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = (H, H, \ldots, H) \tag{4.5}$$

and (cf (1.12))

$$\lim_{N\to\infty} \nu_N = \lim_{N\to\infty} \beta N^{-1} \sum_{i,j=1}^{N} \rho_{ij} = \nu. \tag{4.6}$$

Hence (cf (1.11))

$$z = \lim_{n\to\infty} \left(\sum_{\alpha=1}^{k} (\nu r_\alpha + n^{-1/2}\beta H)^2 + n^{-1} \sum_{\alpha=k+1}^{n} \beta^2 H^2 \right)^{1/2} \tag{4.7}$$

$$= (\nu^2 \|\boldsymbol{r}\|^2 + \beta^2 H^2)^{1/2} \equiv (\nu^2 r^2 + \beta^2 H^2)^{1/2}. \tag{4.8}$$

This formula can be interpreted as saying that for large n the external field in (4.1) is essentially transverse.

The minimum in (4.2) is evaluated straightforwardly. For $\nu > \frac{1}{2} + (\frac{1}{4} + \beta^2 H^2)^{1/2}$ the minimum occurs for $\nu r = (\nu^2 - \nu - \beta^2 H^2)^{1/2}$. For $\nu < \frac{1}{2} + (\frac{1}{4} + \beta^2 H^2)^{1/2}$ the minimum is attained at $r = 0$. It follows that

$$\lim_{n\to\infty} \beta\psi_n(\beta, H) = \begin{cases} \frac{1}{2}[1-\nu+\ln\nu-(\beta^2H^2/\nu)] & \nu > \frac{1}{2}+(\frac{1}{4}+\beta^2H^2)^{1/2} \\ \frac{1}{2}-(\frac{1}{4}+\beta^2H^2)^{1/2}+\frac{1}{2}\ln[\frac{1}{2}+(\frac{1}{4}+\beta^2H^2)^{1/2}] & \nu < \frac{1}{2}+(\frac{1}{4}+\beta^2H^2)^{1/2}. \end{cases} \tag{4.9}$$

With the *correspondence* $\nu = \beta\lambda_1$ it is now easily seen that

$$\lim_{n\to\infty} \psi_n(\beta, H) = \psi(\beta, H) \tag{4.10}$$

with $\psi(\beta, H)$ given by (1.15) and (1.17). This establishes that the spherical model (1.1) and the n-vector model (4.1) are thermodynamically equivalent in the limit $N, n \to \infty$ (k fixed).

It remains to be shown that the n-vector model (1.7), with an anisotropic field, is thermodynamically equivalent in the spherical limit to the spherical model defined by

$$\mathscr{H}' = -\sum_{1\leq i,j\leq k} \rho_{ij}x_ix_j - (N/k)^{1/2}H\sum_{i=1}^{k} x_i. \tag{4.11}$$

To calculate the partition function of this anisotropic spherical model we follow Lieb and Thompson (1969) and first integrate over the variables $x_{k+1}, x_{k+2}, \ldots, x_N$. We then introduce $y_i = N^{-1/2}x_i$ for $i = 1, 2, \ldots, k$, so that for the purposes of calculating the free energy we can take

$$Z'_N = \int_{\Sigma_{i=1}^k y_i^2 \leq 1} \cdots \int \mathrm{d}y_1 \ldots \mathrm{d}y_k \exp\left\{N\left[\beta \sum_{1\leq i<j\leq k} \rho_{ij}x_ix_j + k^{-1/2}B\sum_{i=1}^{k} y_i + \tfrac{1}{2}\ln\left(1-\sum_{i=1}^{k} y_i^2\right)\right]\right\}. \tag{4.12}$$

Here Laplace's method can be applied directly yielding

$$-\beta\psi' = \lim_{N\to\infty} N^{-1}\ln Z'_N$$

$$= \max_{\Sigma_{i=1}^k y_i^2 \leq 1} \left[\tfrac{1}{2}\beta\sum_{i,j=1}^{k} \rho_{ij}y_iy_j + k^{-1/2}B\sum_{i=1}^{k} y_i + \tfrac{1}{2}\ln\left(1-\sum_{i=1}^{k} y_i^2\right)\right] \tag{4.13}$$

$$= \max_{0\leq r\leq 1} [\tfrac{1}{2}\beta\lambda_1 r^2 + Br + \tfrac{1}{2}\ln(1-r^2)]. \tag{4.14}$$

This last step follows from the inequalities

$$\sum_{i,j=1}^{k} \rho_{ij}y_iy_j \leq \lambda_1 \sum_{i=1}^{k} y_i^2 \tag{4.15}$$

and

$$k^{-1/2}\sum_{i=1}^{k} y_i \leq \left(\sum_{i=1}^{k} y_i^2\right)^{1/2}, \tag{4.16}$$

and the observation that the equalities hold if and only if all the y_i are equal.

The maximum in (4.14) occurs for r a solution of the stationary equation ($\nu = \beta\lambda_1$)

$$\nu r^3 + Br^2 + (1-\nu)r - B = 0. \tag{4.17}$$

This is precisely the equation determining r in the limiting n-vector free energy (1.10).

Moreover, writing this equation as

$$\nu r + B = r/(1-r^2) \tag{4.18}$$

it is readily seen that there is exactly one positive root and that this root occurs in the range $0 \leqslant r \leqslant 1$. It is now straightforward to show that

$$\max_{0\leqslant r\leqslant 1} [\tfrac{1}{2}\nu r^2 + Br + \tfrac{1}{2}\ln(1-r^2)]$$

$$= \max_{0\leqslant r\leqslant 1} \tfrac{1}{2}[\![-\nu r^2 - 1 + (1+4z^2)^{1/2} - \ln\{\tfrac{1}{2}[1+(1+4z^2)^{1/2}]\}]\!] \tag{4.19}$$

with

$$z = \nu r + B, \tag{4.20}$$

by using the equation (cf (4.18))

$$r^2 z + r - z = 0. \tag{4.21}$$

This establishes that the free energy of the spherical model (4.11) is of the required limiting n-vector form given by (1.10) and (1.11).

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